On the D3-brane description of some $1/4$ BPS Wilson loops

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Abstract: Recently it has been proposed that Wilson loops in high-dimensional representations in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (or multiply wrapped loops) are described by D-branes in $AdS_5 \times S^5$, rather than by fundamental strings. Thus far explicit D3-brane solutions have been only found in the case of the half-BPS circle or line. Here we present D3-brane solutions describing some $1/4$ BPS loops. In one case, where the loop is conjectured to be given by a Gaussian matrix model, the action of the brane correctly reproduces the expectation value of the Wilson loop including all $1/N$ corrections at large λ. As in the corresponding string solution, here too we find two classical solutions, one stable and one not. The unstable one contributes exponentially small corrections that agree with the matrix model calculation.

Contents

1. Introduction

In the AdS/CFT correspondence [1] Wilson loops are usually described by macroscopic fundamental strings that end along the loop at the holographic boundary of the space [2, 3]. This realizes the string picture of gauge theories, albeit in a conformal, not confining setting. In string theory some objects can metamorphose into others and it was indeed realized in [4] that a multiply wrapped Wilson loop (or a loop in a high-dimensional representation) is better described by a D-brane rather than by a large number of coincident strings.[1]

Initially a D3-brane configuration was found and later also a D5-brane configuration. In both cases they corresponded to a 1/2 BPS Wilson loop, either a straight line or a circle. The D3-brane describes the Wilson loop in the symmetric representation (which, in the strong coupling regime, seems to be dominated by the single-trace multiply wrapped loop) whereas the D5-brane gives the loop in the antisymmetric representation [5 – 10].

Since the initial description of the half-BPS loop in terms of D3-branes, such solutions were not found in any other system. It turned out to be simpler to find D5-brane solutions, and some non-supersymmetric examples were studied [5, 11, 12].

In this paper we find some systems where it is possible to find solutions for the D3-branes starting from first-order equations derived from the supersymmetry conditions. All the examples we present are Wilson loop operators which preserve 1/4 of the supersymmetry generators. First we consider the system of a straight 1/2 BPS Wilson loop with the

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[1]Already in [2] D3-branes were proposed as a possible holographic description of Wilson loops.

insertion of two 1/2 BPS local operators such that the combined system preserves 1/4 of the supercharges. If there was only the Wilson loop, the D3-brane would have been the one of [4], while if only the local operators were present, that would have involved the original giant gravitons [13 – 15].

This combined system of a Wilson loop and a local operator was presented in [16], where it was shown to be supersymmetric and the relevant string solution was found. In section 2 we present the D3-brane solution preserving the same supersymmetries, and interpolating between the "giant Wilson loop" of [4] near the boundary and a giant graviton in the center of AdS_5.

The second system, which will be described in section 3 involves circular Wilson loops which couple to three of the $\mathcal{N} = 4$ scalars. This system, first presented in [17] (as a generalization of an example in [18]) and studied further in [19], also preserves 1/4 of the supersymmetries, but with a different combination of generators than in the previous example. At two loop order in the perturbative expansion the interacting graphs (in the Feynman gauge) cancel, which led to the conjecture that only ladder/rainbow diagrams contribute to these operators. All those diagrams combine nicely into a matrix model which was then compared with the string calculation in $AdS_5 \times S^5$. As we shall review in section 3 below, the results agreed including a subleading term, a world-sheet instanton, which matched a correction to the asymptotic expansion of the matrix model at strong coupling.

The calculation using D3-branes is applicable for a Wilson loop in a symmetric representation whose rank k is of order N. At large λ the analog observable in the matrix model agrees with the single-trace multiply wrapped loop which is given by a function of the ratio k/N and thus the D3-brane calculation captures non-planar corrections to the usual string calculation. We are able to compare the matrix model and the D-brane calculation for arbitrary k/N and find an agreement and a check of the aforementioned conjecture to all orders in $1/N$.

The two systems presented in sections 2 and 3 are quite different. We combined them in the same paper since they both preserve eight supercharges, and some of the technical details of the calculations are similar. Also, the system in section 2 serves as a good warm-up exercise to the more interesting system in section 3. Because of the differences between the two systems, more details on the two setups will be given in each of those sections.

2. Wilson loop with insertions

2.1 Setup

We consider here a Wilson loop operator in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory on $S^3 \times \mathbb{R}$, where the line is the time direction with Lorentzian signature. The loop will be comprised of one line in the time direction along a point on S^3, and another line going in the opposite direction at the antipodal point. In addition we will include the insertion of local operators at the infinite past and infinite future.

Under the exponential map (after Wick-rotation), the space is mapped to flat $\mathbb{R}^4$ and the two lines to a single line through the origin with the local insertions at the origin and

at infinity. Without the insertions this Wilson loop preserves half the supersymmetries of the vacuum, and we will consider the local insertions to also be half-BPS, Z^J, where $Z = \Phi_1 + i\Phi_2$ is a complex scalar field (if Z^J is at the origin the charge has to be absorbed by $\bar{Z}^J$ at infinity). Note that the insertions are not gauge invariant, since they are not traced over, and transform in the adjoint representation of the gauge group. The entire configuration is nonetheless gauge invariant, because of the presence of the Wilson loop. This guarantees that in the string picture the charge generated by the local operator is carried by the open string (or D-brane) representing the Wilson loop, and not by another supergravity field.

Recall that the half-BPS Wilson loop contains also a coupling to one of the scalars. For the combined system to be 1/4 BPS, this scalar has to be orthogonal to Z and $\bar{Z}$, so we take it to be Φ_3. Formally we can write the Wilson loop as

$$W_{Z^J} = \operatorname{Tr} \mathcal{P} \left[Z^J(-\infty)e^{i \int_{-\infty}^{\infty}(A_t(t,0)+\Phi_3(t,0))dt} \bar{Z}^J(\infty)e^{i \int_{\infty}^{-\infty}(A_t(t,\pi)+\Phi_3(t,\pi))dt} \right] . \quad (2.1)$$

The arguments of A_t and Φ_3 are the time and the two points on S^3 given by an angle at 0 and π.

2.2 String solution

This Wilson loop was studied in [16] as were some non-supersymmetric generalizations of it and they were related to a certain spin-chain system. There it was proven that (2.1), which was the pseudo-vacuum of the spin-chain system, is supersymmetric. Also the string solution describing it at large J and large 't Hooft coupling λ was given. We review it here.

Take the following metric for $AdS_5 \times S^2$ (the other directions on S^5 do not play any role and we do not write them explicitly)

$$\frac{ds^2}{L^2} = -\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \left(d\chi^2 + \sin^2 \chi (d\vartheta^2 + \sin^2 \vartheta \, d\varphi^2) \right) + d\theta^2 + \sin^2 \theta d\phi^2 . \quad (2.2)$$

L is the radius of curvature related to the 't Hooft coupling and the string tension by $L^4 = \lambda \, \alpha'^2$. The Wilson loop should reach the boundary at $\chi = 0$ and $\chi = \pi$. At those points it should approach $\theta = 0$ on the S^2, which is the direction corresponding to Φ_3. In the bulk the string should rotate around this sphere carrying the angular momentum related to Z^J.

The solution to the string equations of motion which satisfies these conditions is

$$\phi = t , \qquad \sin \theta = \frac{1}{\cosh \rho} . \quad (2.3)$$

There are two parts to the string: at $\chi = 0$ and at $\chi = \pi$. They are continuously connected to each other beyond $\rho = 0$. For a full derivation of the solution see [16].

Some interesting issues arise when studying the analog system in Euclidean signature. Those were discussed in [20].

2.2.1 Supersymmetry analysis

A precise counting of the supersymmetries preserved by the string solution (2.3) was performed in [16]. Here we briefly review that computation.

The number of supersymmetries preserved by the string is equal to the number of independent solutions to the equation $\Gamma\epsilon = \epsilon$. The κ-symmetry projector Γ is given by

$$\Gamma = \frac{1}{\sqrt{-\det g}}\, \partial_t x^\mu \partial_\rho x^\nu \gamma_\mu \gamma_\nu K\,, \tag{2.4}$$

where g is the induced metric on the world-sheet parameterized by t and ρ, K acts by complex conjugation, and $\gamma_\mu = e_\mu^a \Gamma_a$ with Γ_a constant tangent space gamma-matrices. The dependence of the Killing spinors ϵ on the relevant coordinates of the metric (2.2) is

$$\epsilon = e^{-\frac{i}{2}\rho\,\Gamma_\star\Gamma_1} e^{-\frac{i}{2}t\,\Gamma_\star\Gamma_0} e^{-\frac{i}{2}\theta\,\Gamma_\star\Gamma_5} e^{\frac{1}{2}\phi\,\Gamma_{56}}\epsilon_0\,, \tag{2.5}$$

where $\Gamma_\star = \Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4$ is the product of all gamma-matrices in the AdS_5 directions and ϵ_0 is any constant chiral complex 16-component spinor. The spinors ϵ solve the Killing equation

$$\left(\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\Gamma_{ab} + \frac{i}{2L}\Gamma_\star\gamma_\mu\right)\epsilon = 0\,. \tag{2.6}$$

Inserting the solution (2.3) into the expression (2.4) it is easy to see that Γ does not depend on t. The only place where t appears is in the exponent of the Killing spinors. Since the projection equation has to hold for all t and ρ we eliminate this dependence by imposing the condition

$$\Gamma_\star\Gamma_{056}\epsilon_0 = i\epsilon_0\,, \tag{2.7}$$

so that the Killing spinors become

$$\epsilon = e^{-\frac{i}{2}\rho\,\Gamma_\star\Gamma_1} e^{-\frac{i}{2}\theta\,\Gamma_\star\Gamma_5}\epsilon_0\,. \tag{2.8}$$

After some manipulation the action of the projector can be written as

$$\Gamma\epsilon = -e^{-\frac{i}{2}\rho\,\Gamma_\star\Gamma_1} e^{-\frac{i}{2}\theta\,\Gamma_\star\Gamma_5}\,\Gamma_{01}K\epsilon_0\,, \tag{2.9}$$

so the projector equation is solved by all constant spinors satisfying

$$\Gamma_{01}K\epsilon_0 = -\epsilon_0\,. \tag{2.10}$$

It is easy to verify that the two conditions (2.7) and (2.10) are consistent with each-other, so there are eight linearly independent real solutions to this equation. Thus the string solution preserves $1/4$ of the supersymmetries.

2.3 D3-brane solution

We look now for the D3-brane solution associated to this Wilson loops with insertions. The loop is in the time direction, as reviewed above, and preserves an $SO(3) \times SO(3)$ symmetry, the first being part of the AdS_5 isometry and the other coming from the S^5.

It is convenient to use the metric (2.2) and fix a static gauge where t, ρ, ϑ and φ are the world-volume coordinates on the D3-brane. The ansatz is then

$$\chi = \chi(\rho), \qquad \theta = \theta(\rho), \qquad \phi = t. \tag{2.11}$$

The brane action consists of a Dirac-Born-Infeld (DBI) part and of a Wess-Zumino (WZ) term, which captures the coupling to the background RR form

$$S = T_{D3} \int e^{-\Phi} \sqrt{-\det(g + 2\pi\alpha' F)} - T_{D3} \int P[C_4], \tag{2.12}$$

where $T_{D3} = \frac{N}{2\pi^2 L^4}$ is the brane tension and $P[C_4]$ denotes the pullback of the 4-form to the brane world-volume. The solution should include a non-zero electric field, carrying k units of flux associated to the Wilson loop. Because of the symmetry of the system, it will be in the direction $F_{t\rho}(\rho)$.

With the ansatz above the DBI action reads (in the following we absorb a factor of $2\pi\alpha'/L^2$ in the definition of $F_{t\rho}$)

$$S_{DBI} = \frac{2N}{\pi} \int dt\, d\rho\, \sinh^2\rho \sin^2\chi \sqrt{(\cosh^2\rho - \sin^2\theta)(1 + \sinh^2\rho\,\chi'^2 + \theta'^2) - F_{t\rho}^2}, \tag{2.13}$$

whereas the WZ term is given by

$$S_{WZ} = \frac{2N}{\pi} \int dt\, d\rho\, \sinh^4\rho \sin^2\chi\, \chi', \tag{2.14}$$

and the relative sign between these two terms in the action is positive. In these formulas the $'$ denotes a derivative with respect to ρ.

It is rather complicated to solve the equations of motion coming from this action. Instead of trying to do this, we write down the supersymmetry equations derived from requiring κ-symmetry. These are first-order rather than second-order and can be integrated easily.

The κ-symmetry projector associated with the D3-brane embedding is (see for example [21])

$$\Gamma = \mathcal{L}_{DBI}^{-1} \left(\Gamma_{(4)} + L^2 F_{t\rho} \Gamma_{(2)} K \right) I, \tag{2.15}$$

where K acts by complex conjugation, I by multiplication by $-i$, and

$$\begin{aligned}
\Gamma_{(4)} &= \partial_t x^\mu \partial_\rho x^\nu \partial_\vartheta x^\xi \partial_\varphi x^\zeta \gamma_\mu \gamma_\nu \gamma_\xi \gamma_\zeta = (\gamma_t + \gamma_\phi)(\gamma_\rho + \chi'\,\gamma_\chi + \theta'\,\gamma_\theta)\gamma_\vartheta \gamma_\varphi, \\
\Gamma_{(2)} &= \partial_\vartheta x^\mu \partial_\varphi x^\nu \gamma_\mu \gamma_\nu = \gamma_\vartheta \gamma_\varphi,
\end{aligned} \tag{2.16}$$

with $\gamma_\mu = e_\mu^a \Gamma_a$. Using the vielbeins

$$\begin{aligned}
e^0 &= L\cosh\rho\, dt, \qquad e^1 = L\, d\rho, \qquad e^2 = L\sinh\rho\, d\chi, \\
e^3 &= L\sinh\rho\sin\chi\, d\vartheta, \qquad e^4 = L\sinh\rho\sin\chi\sin\vartheta\, d\varphi, \\
e^5 &= L\, d\theta, \qquad e^6 = L\sin\theta\, d\phi,
\end{aligned} \tag{2.17}$$

and the ansatz (2.11), the projectors $\Gamma_{(4)}$ and $\Gamma_{(2)}$ can be explicitly written as

$$\Gamma_{(4)} = L^2(\cosh\rho\,\Gamma_0 + \sin\theta\,\Gamma_6)(\Gamma_1 + \sinh\rho\,\chi'\,\Gamma_2 + \theta'\,\Gamma_5)\,\Gamma_{(2)}\,,$$
$$\Gamma_{(2)} = L^2\sinh^2\rho\sin^2\chi\sin\vartheta\,\Gamma_{34}\,. \tag{2.18}$$

Adding the dependence on the other coordinates into (2.5), the Killing spinors for the metric (2.2) are

$$\epsilon = e^{-\frac{i}{2}\rho\,\Gamma_\star\Gamma_1}e^{-\frac{i}{2}t\,\Gamma_\star\Gamma_0}e^{\frac{1}{2}\chi\,\Gamma_{12}}e^{\frac{1}{2}\vartheta\,\Gamma_{23}}e^{\frac{1}{2}\varphi\,\Gamma_{34}}e^{-\frac{i}{2}\theta\,\Gamma_\star\Gamma_5}e^{\frac{1}{2}\phi\,\Gamma_{56}}\epsilon_0\,. \tag{2.19}$$

From the supersymmetry analysis in the string case, we know that the constant spinors ϵ_0 satisfy the conditions (2.7) and (2.10)

$$K\epsilon_0 = -\Gamma_{01}\epsilon_0\,, \qquad \Gamma_6\epsilon_0 = -i\Gamma_{12345}\epsilon_0\,. \tag{2.20}$$

Plugging $\phi = t$ in the expression (2.19) and using the second constraint in the equation above, the Killing spinors may be rewritten as

$$\epsilon = e^{-\frac{i}{2}\rho\,\Gamma_\star\Gamma_1}e^{-\frac{i}{2}\theta\,\Gamma_\star\Gamma_5}e^{\frac{1}{2}\chi\,\Gamma_{12}}M\epsilon_0\,, \tag{2.21}$$

where

$$M = e^{\frac{1}{2}\vartheta\,\Gamma_{23}}e^{\frac{1}{2}\varphi\,\Gamma_{34}}\,. \tag{2.22}$$

The differential equations we are looking for will come from considering the projector equation

$$\Gamma\epsilon = \epsilon\,. \tag{2.23}$$

To simplify it, we move the matrix $e^{-\frac{i}{2}\rho\,\Gamma_\star\Gamma_1}e^{-\frac{i}{2}\theta\,\Gamma_\star\Gamma_5}e^{\frac{1}{2}\chi\,\Gamma_{12}}$ to the left of the projector Γ, using some gamma-matrix algebra and applying the constraints (2.20) (note that ϵ_0 and $M\epsilon_0$ satisfy the same constraints). In this way we get a set of 8 differential equations in θ, χ and $F_{t\rho}$ (on the left we indicate the gamma-matrix structure the equations come from)

$$
\begin{aligned}
\Gamma_{0345}:\quad & 0 = F_{t\rho}\sinh\rho\cos\chi\sin\theta - \theta'(\cosh^2\rho - \sin^2\theta) \\
\Gamma_\star\Gamma_5:\quad & 0 = F_{t\rho}\sinh\rho\sin\chi\sin\theta - \chi'\sinh^2\rho\sin\theta\cos\theta \\
\Gamma_{0234}:\quad & 0 = (\cosh^2\rho - \sin^2\theta)\sin\chi + \chi'\cosh\rho\sinh\rho\cos\chi\cos^2\theta \\
\Gamma_{12}:\quad & 0 = F_{t\rho}\sinh\rho\cos\chi\cos\theta + \theta'\sin\theta\cos\theta + \cosh\rho\sinh\rho \\
\Gamma_{15}:\quad & 0 = \chi'\cosh\rho\sinh\rho\cos\chi\sin\theta\cos\theta - \theta'\cosh\rho\sinh\rho\sin\chi + \sin\chi\sin\theta\cos\theta \\
\Gamma_{25}:\quad & 0 = F_{t\rho}\cosh\rho\sin\theta + \cos\chi\sin\theta\cos\theta - \chi'\cosh\rho\sinh\rho\sin\chi\sin\theta\cos\theta \\
 & \quad -\theta'\cosh\rho\sinh\rho\cos\chi \\
\Gamma_{0134}:\quad & 0 = F_{t\rho}\cosh\rho\cos\theta + (\cosh^2\rho - \sin^2\theta)\cos\chi - \chi'\cosh\rho\sinh\rho\sin\chi\cos^2\theta \\
1:\quad & 1 = -L^4\mathcal{L}_{DBI}^{-1}\sinh^2\rho\sin^2\chi\sin\vartheta\left(F_{t\rho}\sinh\rho\sin\chi\cos\theta + \chi'\sinh^2\rho\sin^2\theta\right)
\end{aligned}
\tag{2.24}
$$

One can solve for θ', χ' and $F_{t\rho}$ using for instance the first three equations. Once these are solved, the remaining five are automatically satisfied. The first three equations give

$$\theta' = -\tan\theta\tanh\rho\,,$$
$$\chi'\cot\chi = -\frac{\cosh^2\rho - \sin^2\theta}{\cosh\rho\sinh\rho\cos^2\theta}\,, \tag{2.25}$$
$$F_{t\rho} = -\frac{\cosh^2\rho - \sin^2\theta}{\cosh\rho\cos\theta\cos\chi}\,.$$

The solution to the first equation is

$$\sin \theta = \frac{C_1}{\cosh \rho}. \tag{2.26}$$

The integration constant C_1 is related (in a complicated way) to the amount of angular momentum carried by the brane. After plugging the solution for θ into the expressions for χ' and $F_{t\rho}$, we find

$$\chi' \cot \chi = -\frac{\cosh^4 \rho - C_1^2}{\sinh \rho \cosh \rho (\cosh^2 \rho - C_1^2)}, \tag{2.27}$$

which is solved by

$$\sin \chi = C_2 \frac{\coth \rho}{\sqrt{\cosh^2 \rho - C_1^2}}, \tag{2.28}$$

with C_2 a second integration constant. Finally the electric field is

$$F_{t\rho} = -\frac{\cosh^4 \rho - C_1^2}{\cosh^2 \rho \sqrt{\cosh^2 \rho - C_1^2 - C_2^2 \coth^2 \rho}}. \tag{2.29}$$

Plugging the BPS equations (2.25) into the DBI action (2.13), the square root simplifies to

$$\sqrt{(\cosh^2 \rho - \sin^2 \theta)(1 + \sinh^2 \rho \, \chi'^2 + \theta'^2) - F_{t\rho}^2} = (\cosh^2 \rho - \sin^2 \theta)\frac{\tanh \rho \tan \chi}{\cos^2 \theta}. \tag{2.30}$$

It is then straightforward to check that the solutions (2.26), (2.28) and (2.29) satisfy the brane equations of motion stemming from (2.13) and (2.14).

2.3.1 Conserved charges

The solution has two integration constants C_1 and C_2 which are related to the two conserved charges carried by the brane: the rank of the symmetric representation (or the number of windings of the Wilson loop) k, and the angular momentum J around the S^2 in the S^5.

The first charge k is the conjugate momentum to the gauge field after integrating over ϑ and φ

$$k = \Pi = \frac{2\pi \alpha'}{L^2} T_{D3} \int d\vartheta \, d\varphi \, \frac{\delta \mathcal{L}}{\delta F_{t\rho}} = \frac{4N}{\sqrt{\lambda}} C_2. \tag{2.31}$$

Then

$$C_2 = \frac{k\sqrt{\lambda}}{4N} \equiv \kappa. \tag{2.32}$$

If we take $\kappa \to 0$ we recover the string solution (2.3). Notice that the electric field $F_{t\rho}$ does not vanish in this limit.

The other conserved charge carried by the brane is the angular momentum J

$$
\begin{aligned}
J &= 2 T_{D3} \int d\vartheta \, d\varphi \, d\rho \, \frac{\delta \mathcal{L}}{\delta \dot{\phi}} \\
&= -\frac{4N}{\pi} \int d\rho \, \frac{\sinh^2 \rho \sin^2 \chi \sin^2 \theta \, (1 + \sinh^2 \rho \, \chi'^2 + \theta'^2)}{\sqrt{(\cosh^2 \rho - \sin^2 \theta)(1 + \sinh^2 \rho \, \chi'^2 + \theta'^2) - F_{t\rho}^2}}.
\end{aligned}
\tag{2.33}
$$

Here the range of the ρ integral is $[\text{arccosh}\, C_1, \infty)$, which like in the case of the string, covers only half the world-volume, with $\chi < \pi/2$. A multiplicative factor of 2 was included to account for the other branch with $\chi > \pi/2$.

Plugging the explicit solutions in this expression it is easy to see that $J \to 0$ when $C_1 \to 0$. In this limit the brane does not rotate along the S^2 and the solution reduces to

$$
\begin{aligned}
\sin \theta &= 0 \,, \\
\sin \chi \, \sinh \rho &= \kappa \,, \\
F_{t\rho} &= -\frac{\cosh \rho}{\cos \chi} \,,
\end{aligned}
\tag{2.34}
$$

which is, after a conformal transformation, the same as the 1/2 BPS brane of [4].

The energy gets contributions from the DBI action and the Wess-Zumino term

$$
\begin{aligned}
E_{DBI} &= 2\, T_{D3} \int d\vartheta \, d\varphi \, d\rho \, \frac{\delta \mathcal{L}_{DBI}}{\delta \dot{t}} \\
&= \frac{4N}{\pi} \int d\rho \, \frac{\sinh^2 \rho \sin^2 \chi \cosh^2 \rho \, (1 + \sinh^2 \rho \, \chi'^2 + \theta'^2)}{\sqrt{(\cosh^2 \rho - \sin^2 \theta)(1 + \sinh^2 \rho \, \chi'^2 + \theta'^2) - F_{t\rho}^2}} \,, \\
E_{WZ} &= 2\, T_{D3} \int d\vartheta \, d\varphi \, d\rho \, \frac{\delta \mathcal{L}_{WZ}}{\delta \dot{t}} = \frac{4N}{\pi} \int d\rho \, \sinh^4 \rho \sin^2 \chi \, \chi' \,.
\end{aligned}
\tag{2.35}
$$

In addition one has to add a total derivative term, which serves as a Legendre transform from the gauge field coordinate to the conjugate momentum Π, which is the correct canonical variable in this problem [4]. This is

$$
E_{L.T.} = \frac{2L^2}{2\pi \alpha'} \int d\rho \, \Pi \, F_{t\rho} = -\frac{4N}{\pi} \kappa \int d\rho \, \frac{\cosh^2 \rho - \sin^2 \theta}{\cosh \rho \cos \theta \cos \chi} \,.
\tag{2.36}
$$

Plugging the BPS equations in the formulas (2.33), (2.35) and (2.36) above, one can see that

$$
E_{DBI} + E_{WZ} + E_{L.T.} + J = \frac{4N}{\pi} \int d\rho \, \frac{\cosh^2 \rho - \sin^2 \theta}{\cosh \rho \cos \chi} \left[\sinh \rho \sin \chi - \frac{\kappa}{\cos \theta} \right] \,.
\tag{2.37}
$$

Using the explicit solution it is easy to check that the term in square brackets vanishes, so we get $E = -J = |J|$.

3. Wilson loop wrapping a circle on S^5

3.1 Setup

In this section we shall look at a family of circular Wilson loops that couple to three of the six scalars of the $\mathcal{N} = 4$ multiplet. These operators were presented in [17] and studied in detail in [19]. As for the loop in the previous section, the S^5 part of the gravity calculation will reduce to an S^2 subspace. The difference will be that here the couplings to the scalars are smeared around the loop and not localized at two points.

While the Wilson loop follows a curve on the boundary of AdS_5 parameterized by

$$x^1 = R\cos s\,, \qquad x^2 = R\sin s\,, \tag{3.1}$$

the scalar to which it couples is given by the linear combination

$$\Phi(s) = \Phi_3\cos\theta_0 + \sin\theta_0(\Phi_1\cos s + \Phi_2\sin s)\,, \tag{3.2}$$

with an arbitrary fixed parameter θ_0. The loop may be written (in Euclidean signature) as

$$W_{\theta_0} = \operatorname{Tr}\mathcal{P}\exp\left[\oint\left(iA_\mu(s)\dot{x}^\mu + |\dot{x}|\Phi(s)\right)ds\right]\,. \tag{3.3}$$

In the special case of $\theta_0 = 0$ this is the usual half-BPS circle, while for $\theta_0 = \pi/2$ this is a special case of the supersymmetric Wilson loops constructed by Zarembo [18].

It was shown in [19] that up to order $(g_{YM}^2 N)^2$ all interacting graphs in the Feynman gauge cancel and the only contribution comes from ladder diagrams where the propagator is a constant proportional to $\cos^2\theta_0$. This naturally led to the conjecture that the expectation value of this Wilson loop is given by the same matrix model as the half-BPS one [22, 23] with the replacement of the coupling λ by $\lambda' = \lambda\cos^2\theta_0$. This gives the prediction

$$\langle W_{\theta_0}\rangle = \frac{1}{N}L_{N-1}^1\left(-\frac{\lambda'}{4N}\right)\exp\left[\frac{\lambda'}{8N}\right]\,, \tag{3.4}$$

where L_{N-1}^1 is a Laguerre polynomial. In [19] only the planar limit of this expression

$$\langle W_{\theta_0,\,\text{planar}}\rangle = \frac{2}{\sqrt{\lambda'}}I_1\left(\sqrt{\lambda'}\right)\,, \tag{3.5}$$

was considered (I_1 is a modified Bessel function). String theory provided exact agreement with the strong coupling expansion of this expression, as we shall review shortly. Furthermore, the same rescaling was observed in the computation of correlation functions between this 1/4 BPS loop and chiral primary operators [24].

In the present calculation we want to capture a different limit, beyond the planar one. We consider a multiply wrapped Wilson loop, or a loop in the k-th symmetric representation[2], keeping the quantity $\kappa' \equiv k\sqrt{\lambda'}/4N$ fixed while taking both N and λ to infinity. This is the limit that was discussed in [4, 6, 8, 9] for the half-BPS loop, and in this limit the matrix model reduces to

$$\langle W_{\kappa'}\rangle = \exp\left[2N\left(\kappa'\sqrt{1+\kappa'^2} + \operatorname{arcsinh}\kappa'\right)\right]\,. \tag{3.6}$$

There is also a subleading contribution, that we did not include in the formula above, obtained by replacing $\kappa' \to -\kappa'$. The appearance of this term can be explained by the fact that perturbation theory should be invariant under $\lambda' \to e^{2i\pi}\lambda'$. At strong coupling the expectation value of the Wilson loop depends on $\sqrt{\lambda'}$, so that an extra term with

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[2]It has been proven in [8] and [9] that in the matrix model the multiply wound loop $W^{(k)}$ and the totally symmetric operator W_{S_k} coincide in the strong coupling regime.

$\kappa' \to -\kappa'$ is needed. In the planar approximation this subleading term reduces to $e^{-\sqrt{\lambda'}}$, which appears in the large λ' expansion of the Bessel function in (3.5).

Later in this section we will be able to construct a D3-brane which is dual to the multiply wrapped 1/4 BPS Wilson loop and we will recover (3.6) from supergravity. This computation will also produce the subleading contribution discussed above, which will correspond to an unstable D3-brane solution.

3.2 String solution

To write the relevant string solutions in the dual supergravity picture we use the following metric on $AdS_5 \times S^2$ (as in the previous example we drop the directions on S^5 which do not play a role here)

$$\frac{ds^2}{L^2} = -d\chi^2 + \cos^2\chi(d\rho^2 + \sinh^2\rho\, d\psi^2) + \sin^2\chi(d\sigma^2 + \sinh^2\sigma\, d\varphi^2) + d\theta^2 + \sin^2\theta d\phi^2\,. \quad (3.7)$$

This metric has Lorentzian signature, which is somewhat more natural for the supersymmetry analysis, but later we will also use the Euclidean version obtained by Wick rotating $\chi \to iu$ and $\sigma \to i\vartheta$

$$\frac{ds^2}{L^2} = du^2 + \cosh^2 u(d\rho^2 + \sinh^2\rho\, d\psi^2) + \sinh^2 u(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2) + d\theta^2 + \sin^2\theta d\phi^2\,. \quad (3.8)$$

Note that in the Lorentzian case the χ coordinate foliates AdS_5 by $\mathbb{H}_2 \times \mathbb{H}_2$ surfaces ($\mathbb{H}_2$ is the two-dimensional hyperbolic space, or Euclidean AdS_2), while in the Euclidean case u foliates it into $\mathbb{H}_2 \times S^2$ surfaces.

The string describing the Wilson loop (3.3) will be at $\chi = 0$ (or $u = 0$) and should end at $\rho \to \infty$ along a circle parameterized by ψ. As we go along this circle we should also move along a circle on S^2, the parallel at angle θ_0 spanned by the angle ϕ. We take the ansatz where along the entire world-sheet we equate ψ and ϕ. As mentioned, the asymptotic value of θ should be θ_0. In [19] two solutions with these boundary conditions were found

$$\phi = \psi\,, \qquad \sinh\rho(\sigma) = \frac{1}{\sinh\sigma}\,, \qquad \sin\theta = \frac{1}{\cosh(\sigma_0 \pm \sigma)}\,. \quad (3.9)$$

Here σ is a world-sheet coordinate and σ_0 is related to the boundary value of θ by

$$\sin\theta_0 = \frac{1}{\cosh\sigma_0}\,. \quad (3.10)$$

One can eliminate σ from the previous equations to find the relation

$$\cosh\rho\cos\theta\sin\theta_0 - \sinh\rho\sin\theta\cos\theta_0 = \pm\sin\theta_0\,. \quad (3.11)$$

The two sign choices correspond to surfaces extending over the north and south pole of S^2 respectively. The classical action for the two cases is equal to

$$S = \mp\cos\theta_0\sqrt{\lambda} = \mp\sqrt{\lambda'}\,. \quad (3.12)$$

The dominant contribution has negative action and corresponds to the surface extended over less than half a sphere. That solution is stable, while the one extending over the other pole has positive action and three unstable modes.

These two solutions were interpreted in [19] as corresponding to the two saddle points in the asymptotic expansion of the Bessel function (3.5)

$$\langle W_{\theta_0, \text{planar}} \rangle \xrightarrow[\lambda' \to \infty]{} \frac{\sqrt{2}}{\sqrt{\pi} \lambda'^{3/4}} \left[e^{\sqrt{\lambda'}} \left(1 + \mathcal{O}(1/\sqrt{\lambda'}) \right) - i e^{-\sqrt{\lambda'}} \left(1 + \mathcal{O}(1/\sqrt{\lambda'}) \right) \right] . \quad (3.13)$$

Furthermore, it was shown there that considering the limit of large λ, while keeping small λ' and integrating over the three modes that are massless for $\lambda' = 0$, yields an identical result to the full planar expression from the matrix model (3.5), including all α' corrections.

The counting of the supersymmetries for the solutions (3.9) goes very similarly to the counting presented in section 2 for the loop with insertions. The dependence of the Killing spinors on the relevant components of the metric (3.7) is[3]

$$\epsilon = e^{-\frac{i}{2} \rho \, \Gamma_\star \Gamma_1} e^{\frac{1}{2} \psi \, \Gamma_{12}} e^{-\frac{i}{2} \theta \, \Gamma_\star \Gamma_5} e^{\frac{1}{2} \phi \, \Gamma_{56}} \epsilon_0 , \quad (3.14)$$

while the constraints analogous to (2.7) and (2.10) are now

$$(\Gamma_{12} + \Gamma_{56}) \epsilon_0 = 0 , \quad (3.15)$$

and

$$K \epsilon_0 = - \left(\cos \theta_0 \, \Gamma_{12} + \sin \theta_0 \Gamma_{16} \right) \epsilon_0 . \quad (3.16)$$

The two conditions (3.15) and (3.16) are compatible and therefore the two string solutions (3.9) preserve one quarter of the supersymmetries, as does the operator W_{θ_0} in the dual gauge theory.

3.3 D3-brane solution

We now move on to the construction of the 1/4 BPS D3-brane which describes the circular Wilson loop in the k-th symmetric representation $W_{\kappa'}$. The supersymmetry analysis will be presented in Lorentzian signature (3.7) to avoid defining the Killing spinors in Euclidean space. The resulting brane has extra factors of i in the projector equations and an over-critical electric field. Moreover it does not seem to correspond to a Wilson loop operator in the gauge theory, but to a higher-dimensional observable. Still we find this way of performing the calculation useful. After presenting the solution we will switch to Euclidean signature (3.8), where the solution will not suffer from those problems and will be perfectly well defined.

We parameterize the brane world-volume by $\{\rho, \psi, \sigma, \varphi\}$. The 1/2 BPS brane has constant $\chi = \arcsin \kappa$ and $\theta = 0$. A natural ansatz for the 1/4 BPS brane is then to take $\chi = \chi(\rho)$, $\theta = \theta(\rho)$ and identify ψ with ϕ. This is consistent with the symmetries of the loop. The asymptotic value of θ at $\rho = \infty$ should be θ_0. To carry the k units of flux represented by the Wilson loop operator, we switch on an electric field $F_{\rho\psi}(\rho)$.

---

[3]This is presented in greater detail in the next subsection (3.18) together with the vielbeins (3.17).

Note that the dependence of θ and χ on ρ explicitly breaks the AdS_2 isometry. This fact makes it difficult to guess a simple ansatz for the solution to the equations of motion of the brane. As in section 2 we will then proceed by looking at the first-order supersymmetry equations which follow from requiring κ-symmetry on the brane world-volume.

We begin by constructing the Killing spinors associated to the $AdS_5 \times S^2$ metric (3.7) with Lorentzian signature. The vielbeins relevant for the D3-brane solution are

$$e^0 = L\, d\chi\,, \qquad e^1 = L\cos\chi\, d\rho\,, \qquad e^2 = L\cos\chi\sinh\rho\, d\psi\,,$$
$$e^3 = L\sin\chi\, d\sigma\,, \qquad e^4 = L\sin\chi\sinh\sigma\, d\varphi\,, \tag{3.17}$$
$$e^5 = L\, d\theta\,, \qquad e^6 = L\sin\theta\, d\phi\,.$$

Using the same notation of section 2, the Killing spinors may then be written as (adding the dependence on ϑ and φ to (3.14))

$$\epsilon = e^{-\frac{i}{2}\chi\Gamma_*\Gamma_0} e^{-\frac{i}{2}\rho\Gamma_*\Gamma_1} e^{\frac{1}{2}\psi\Gamma_{12}} e^{-\frac{1}{2}\sigma\Gamma_{03}} e^{\frac{1}{2}\varphi\Gamma_{34}} e^{-\frac{i}{2}\theta\,\Gamma_*\Gamma_5} e^{\frac{1}{2}\phi\,\Gamma_{56}} \epsilon_0\,. \tag{3.18}$$

The DBI Lagrangian reads (with the sign in the square root appropriate for a brane with Euclidean world-volume and with $F_{\rho\psi}$ containing a factor of $2\pi\alpha'/L^2$)

$$\mathcal{L}_{DBI} = L^4 \sin^2\chi\sinh\sigma\sqrt{(-\chi'^2 + \theta'^2 + \cos^2\chi)(\cos^2\chi\sinh^2\rho + \sin^2\theta) + F_{\rho\psi}^2}\,, \tag{3.19}$$

and the projector associated with the D3-brane is

$$\Gamma = \mathcal{L}_{DBI}^{-1}\left(i\Gamma_{(4)} - L^2 F_{\rho\psi}\Gamma_{(2)}K\right)I\,, \tag{3.20}$$

where, again, K acts by complex conjugation, I by multiplication by $-i$ and

$$\Gamma_{(4)} = (\gamma_\rho + \chi'\,\gamma_\chi + \theta'\gamma_\theta)(\gamma_\psi + \gamma_\phi)\,\Gamma_{(2)}$$
$$= L^2(\cos\chi\Gamma_1 + \chi'\Gamma_0 + \theta'\Gamma_5)(\cos\chi\sinh\rho\Gamma_2 + \sin\theta\Gamma_6)\,\Gamma_{(2)} \tag{3.21}$$
$$\Gamma_{(2)} = \gamma_\sigma\gamma_\varphi = L^2\sin^2\chi\sinh\sigma\,\Gamma_{34}\,.$$

Note that the projector Γ does not depend on ψ. As for the string case we can eliminate the dependence on ψ in the projection equation by imposing

$$(\Gamma_{12} + \Gamma_{56})\epsilon_0 = 0\,, \tag{3.22}$$

and then we impose also the condition

$$K\epsilon_0 = -(\cos\theta_0\Gamma_{12} + \sin\theta_0\Gamma_{16})\epsilon_0\,, \tag{3.23}$$

which both follow from the analysis of the supersymmetries of the string (3.16). The brane solution will then preserve the same quarter of supersymmetries as the string and as the gauge theory observable.

Because of the isometry of the system the factor of

$$M \equiv e^{-\frac{1}{2}\sigma\Gamma_{03}} e^{\frac{1}{2}\varphi\Gamma_{34}} \tag{3.24}$$

commutes with those two constraints, so ϵ_0 and $M\epsilon_0$ satisfy the same conditions.

Using some gamma-matrix algebra and applying the constraints above, we move the matrix $e^{-\frac{i}{2}\chi\,\Gamma_\star\Gamma_0}e^{-\frac{i}{2}\rho\,\Gamma_\star\Gamma_1}e^{-\frac{i}{2}\theta\,\Gamma_\star\Gamma_5}$ to the left of Γ in the projection equation. In this way one gets a set of 8 first order differential equations for θ, χ and $F_{\rho\psi}$ (indicating which gamma-matrix combination leads to them)

$$
\begin{aligned}
\Gamma_{0234}:\quad & 0 = iF_{\rho\psi}\sin\chi\sin\theta\sin\theta_0 + \chi'\sinh\rho(\cos^2\chi - \sin^2\theta) + \cosh\rho\sin\chi\cos\chi\sin^2\theta \\
\Gamma_\star\Gamma_5:\quad & 0 = iF_{\rho\psi}\sin\chi\sin\theta\cos\theta_0 - \chi'\cosh\rho\sin\theta\cos\theta + \sinh\rho\sin\chi\cos\chi\sin\theta\cos\theta \\
\Gamma_{1234}:\quad & 0 = iF_{\rho\psi}\sinh\rho\cos\chi\sin\theta\sin\theta_0 + iF_{\rho\psi}\cosh\rho\cos\chi\cos\theta\cos\theta_0 - \\
& \quad -\chi'\sinh^2\rho\sin\chi\cos\chi - \theta'\sin\theta\cos\theta + \sinh\rho\cosh\rho\cos^2\chi \\
\Gamma_{2345}:\quad & 0 = iF_{\rho\psi}\sinh\rho\cos\chi\sin\theta\cos\theta_0 - iF_{\rho\psi}\cosh\rho\cos\chi\cos\theta\sin\theta_0 - \\
& \quad -\theta'\sinh\rho\cosh\rho\cos^2\chi - \cos^2\chi\sin\theta\cos\theta \\
\Gamma_{01}:\quad & 0 = iF_{\rho\psi}\sinh\rho\cos\chi\cos\theta\cos\theta_0 + iF_{\rho\psi}\cosh\rho\cos\chi\sin\theta\sin\theta_0 - \\
& \quad -\chi'\sinh\rho\cosh\rho\sin\chi\cos\chi + \cos^2\chi(\sinh^2\rho + \sin^2\theta) \\
\Gamma_{05}:\quad & 0 = iF_{\rho\psi}\sinh\rho\cos\chi\cos\theta\sin\theta_0 - iF_{\rho\psi}\cosh\rho\cos\chi\sin\theta\cos\theta_0 \\
& \quad +\theta'(\sinh^2\rho\cos^2\chi + \sin^2\theta) \\
\Gamma_{15}:\quad & 0 = iF_{\rho\psi}\sin\chi\cos\theta\sin\theta_0 - \chi'\sinh\rho\sin\theta\cos\theta + \\
& \quad +\theta'\sinh\rho\sin\chi\cos\chi + \cosh\rho\sin\chi\cos\chi\sin\theta\cos\theta \\
1:\quad & 1 = -iL^4\mathcal{L}_{DBI}^{-1}\sin^2\chi\sinh\sigma\left(iF_{\rho\psi}\sin\chi\cos\theta\cos\theta_0 + \chi'\cosh\rho\sin^2\theta \right. \\
& \quad \left. - \sinh\rho\sin\chi\cos\chi\sin^2\theta\right)
\end{aligned}
$$

(3.25)

The half-BPS solution

$$
\chi' = 0\,,\qquad F_{\rho\psi} = i\cos\chi\sinh\rho
\tag{3.26}
$$

can be recovered by setting $\theta = \theta_0 = 0$ in (3.25).

The equations (3.25) are all consistent with each other. One can solve any three of them, the remaining ones being automatically satisfied. The first three, for example, lead to the equations

$$
\theta' = A\cos^2\chi\cos\theta\,,\qquad \chi' = A\sin\chi\cos\chi\sin\theta\,,
\tag{3.27}
$$

$$
F_{\rho\psi} = -i\frac{\cos\chi\cos\theta}{\cos\theta_0}(A\cosh\rho\sin\theta - \sinh\rho)\,,
\tag{3.28}
$$

where

$$
A = \frac{\sinh\rho\cos\theta\sin\theta_0 - \cosh\rho\sin\theta\cos\theta_0}{(\cos^2\chi - \sin^2\theta)\sinh\rho\cos\theta_0 + \cosh\rho\sin\theta\cos\theta\sin\theta_0}\,.
\tag{3.29}
$$

Taking the ratio of θ' and χ' yields

$$
\sin\chi\cos\theta = C\,,
\tag{3.30}
$$

where C is an integration constant. Inserting this solution into the expression for θ' and solving the resulting differential equation gives

$$
\cos\chi\,(\cosh\rho\cos\theta\sin\theta_0 - \sinh\rho\sin\theta\cos\theta_0) = D\,.
\tag{3.31}
$$

On the AdS_5 side of the ansatz at $\rho = 0$ the circle parameterized by ψ shrinks to a point. For the solution not to be singular at that point, the same has to happen also on the

S^5 side, since $\phi = \psi$. The solution will be regular at $\rho = 0$ only if at that point $\sin \theta = 0$, which then gives D in terms of θ_0 and C as

$$D = \pm \sin \theta_0 \sqrt{1 - C^2}\,, \tag{3.32}$$

where the $+$, $-$ signs correspond respectively to taking either $\theta = 0$ or $\theta = \pi$ at $\rho = 0$, or, in other words, to wrapping the brane around the northern or the southern hemisphere of S^2. Notice that in the string limit ($\chi \to 0$, or $C \to 0$) the expression (3.31) reduces to the string solution (3.11).

These solutions in Lorentzian space are unphysical. The world-volume of the brane is Euclidean, but the electric field is over-critical, leading to an imaginary action. Furthermore, the branes do not end along curves on the boundary, but along higher-dimensional surfaces, and do not provide a holographic description of Wilson loops.

Therefore we analytically continue those solutions to Euclidean signature, where the resulting branes will provide a good holographic dual of Wilson loop operators. We take the Wick rotation

$$\chi = i\,u\,, \qquad \sigma = i\,\vartheta\,. \tag{3.33}$$

In these coordinates, the Euclidean AdS_5 is written as an $\mathbb{H}_2 \times S^2$ fibration as in eq. (3.8). The solution (3.30) and (3.31) in Lorentzian signature now becomes

$$\sinh u \cos \theta = c\,, \tag{3.34}$$

and

$$\cosh u \left(\cosh \rho \cos \theta \sin \theta_0 - \sinh \rho \sin \theta \cos \theta_0 \right) = d\,. \tag{3.35}$$

Similarly to the Lorentzian case the solution is smooth at $\rho = 0$ only for

$$d = \pm \sin \theta_0 \sqrt{1 + c^2}\,. \tag{3.36}$$

The implicit equation (3.35) is solved for ρ as a function of θ by

$$
\begin{aligned}
\sinh \rho &= \mathrm{sign}(\theta_0 - \theta) \frac{\sin \theta \sin \theta_0 \left(\sqrt{1 + c^2} \cos \theta_0 + \cos \theta \sqrt{1 + \frac{c^2 \cos^2 \theta_0}{\cos^2 \theta}} \right)}{\cosh u (\cos^2 \theta - \cos^2 \theta_0)} \\
\cosh \rho &= \mathrm{sign}(\theta_0 - \theta) \frac{\sqrt{1 + c^2} \cos \theta \sin^2 \theta_0 + \sin^2 \theta \cos \theta_0 \sqrt{1 + \frac{c^2 \cos^2 \theta_0}{\cos^2 \theta}}}{\cosh u (\cos^2 \theta - \cos^2 \theta_0)}\,.
\end{aligned}
\tag{3.37}
$$

The sign function allows us to write in a single expression the two solutions corresponding to a brane wrapping over the north or south poles of the S^2. We will assume, without loss of generality, that $\theta_0 \le \pi/2$.

Given that the solution may be written explicitly as a function of θ, it makes sense to use it, instead of ρ, as one of the world-volume coordinate. Thus the world-volume is parameterized by $\{\theta, \psi, \vartheta, \varphi\}$ and $\rho = \rho(\theta)$ and $u = u(\theta)$ are given by the solutions above. This parametrization will be singular in the 1/2 BPS limit, where $\theta = 0$, but that solution is very simple, with arbitrary ρ and constant $u = \mathrm{arcsinh}\, c$.

The DBI action in this signature reads

$$S_{DBI} = 4N \int d\theta \, \sinh^2 u \, \sqrt{(\cosh^2 u \, \rho'^2 + u'^2 + 1)(\cosh^2 u \sinh^2 \rho + \sin^2 \theta) + F_{\theta\psi}^2}\,, \quad (3.38)$$

while the Wess-Zumino term can be written as

$$S_{WZ} = 4N \int d\theta \, \rho' \sinh \rho \left(\frac{u}{2} - \frac{1}{2} \sinh u \cosh u - \sinh^3 u \cosh u \right). \quad (3.39)$$

To obtain these expressions we have integrated over ψ and S^2. Now the $'$ stands for the derivative with respect to θ. We have checked that the solutions found above satisfy the equations of motion coming from (3.38) and (3.39).

In figure 1 we have plotted ρ and u as functions of θ for a D3-brane solution and for comparison also ρ for the analog string solution (in which case $u = 0$). There are two solution, both reaching infinite ρ at θ_0 (for the example pictured we took the values $\theta_0 = \pi/3$ and $\kappa = 1$). The stable solution then goes to $\rho = 0$ at $\theta = 0$, while the unstable solution goes to $\rho = 0$ at $\theta = \pi$.

Note that in the case of the unstable D3-brane solution the coordinate u diverges at the equator $\theta = \pi/2$, as can also be seen from (3.34). This means that the D3-brane reaches the boundary of AdS at that point, and gets reflected back into the interior (after changing the sign of u). One could choose to truncate the surface there and consider either half of the solution. But in the dual gauge theory that will not correspond to a Wilson loop vacuum expectation value. Rather, the D3-brane extending from θ_0 to $\theta = \pi/2$ will be the correlator between the Wilson loop and a two-dimensional surface operator located where the brane reaches the boundary (the surface spanned by $\{\vartheta, \varphi\}$, the radius of the ψ circle shrinks to a point there). The other part of the solution, from $\theta = \pi/2$ to $\theta = \pi$ is the vacuum expectation value of the surface operator itself, with no Wilson loop insertion.

As usual the BPS equations simplify the square root in the DBI action, which in this case reduces to

$$S_{DBI} = 4N \int d\theta \, \left| F_{\theta\psi} \frac{\cos \theta_0}{\cos \theta} \right| \sinh^3 u\,. \quad (3.40)$$

This fact can be used to check the conservation of Π, the momentum conjugate to the gauge field A_ψ

$$\Pi = -i \frac{2\pi\alpha'}{L^2} T_{D3} \int d\vartheta \, d\varphi \frac{\delta \mathcal{L}_{DBI}}{\delta F_{\theta\psi}} = \pm \frac{4N}{\sqrt{\lambda}} \left| \frac{c}{\cos \theta_0} \right| \equiv \pm k, \quad (3.41)$$

where the two signs correspond to the two solutions. This implies that

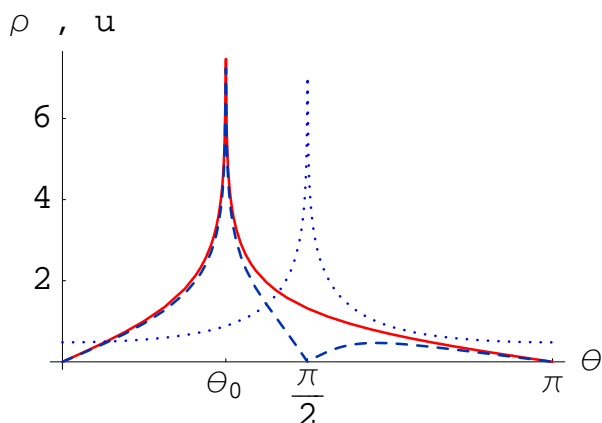$$|c| = \kappa |\cos \theta_0|. \quad (3.42)$$

To compute the full on-shell action we have to supplement the DBI and WZ bulk contributions with total derivatives and boundary terms associated to the electric field $F_{\theta\psi}$ and the scalar $\rho(\theta)$. These are given respectively by[4]

$$S_{L.T.} = -i \frac{L^2}{2\pi\alpha'} \int d\theta \, d\psi \, \Pi \, F_{\theta\psi}\,, \quad (3.43)$$

---

[4]For a general discussion on the role of boundary terms see [25] and [4].

Figure 1: A depiction of string and D3-brane solutions. The solid line gives ρ as a function of θ for the string with boundary value of $\theta = \pi/3$. The D3-brane solution is represented by the dashed and dotted lines which are respectively ρ and u as functions of θ (for $\kappa = 1$). In both cases there are two solutions, a stable one with $0 \leq \theta \leq \theta_0$ (where ρ for the string and D3-brane are nearly indistinguishable) and an unstable one, with $\theta \leq \theta_0 \leq \pi$. The unstable D3-brane solution reaches the boundary of AdS not only at θ_0, but also at $\theta = \pi/2$, where u diverges, but then it turns back and closes smoothly on itself.

and

$$-P_\rho\Big|_{\theta \to \theta_0} = -T_{D3} \int d\psi \, d\vartheta \, d\varphi \, \frac{\delta(\mathcal{L}_{DBI} + \mathcal{L}_{WZ})}{\delta \rho'}\Big|_{\theta \to \theta_0} . \tag{3.44}$$

where P_ρ is the momentum conjugate to ρ.

The boundary term for ρ can be motivated as follows. Let us consider the AdS_5 metric in the Poincaré patch

$$ds^2 = \frac{1}{z^2}(dz^2 + dr_1^2 + r_1^2 d\varphi_1^2 + dr_2^2 + r_2^2 d\varphi_2^2) . \tag{3.45}$$

The transformation relating z to our coordinates is

$$z = \frac{1}{\cosh u \cosh \rho - \cos \vartheta \sinh u} , \tag{3.46}$$

so that the $\rho \to \infty$ region corresponds to $z = 0$. In the Poincaré patch the boundary term associated to z has the form of a Legendre transform evaluated at the boundary of AdS_5 [25, 4]

$$-\int z \, p_z \Big|_{z \sim 0} . \tag{3.47}$$

Using (3.46) it is immediate to verify that in proximity of the boundary $z \, p_z \sim p_\rho$. This justifies the form of the boundary term for ρ.

Now we can evaluate the on-shell action. The bulk and boundary contributions diverge as we approach the boundary of AdS_5, i.e. in the limit $\theta \to \theta_0$. We can regularize these divergences by introducing a cut-off at $\theta_0 - \epsilon$. This leads to the following expression for

the regularized DBI action

$$S_{DBI} = T_{D3} \int_0^{\theta_0 - \epsilon} d\theta \int d\psi \, d\vartheta \, d\varphi \, \mathcal{L}_{DBI} = 4N\kappa^3 \sin\theta_0 \frac{\sqrt{1+c^2}}{\epsilon}$$

$$+ 2N\kappa^3 \frac{\sin\theta_0 \tan\theta_0}{\sqrt{1+c^2}} - N\kappa \sec^2\theta_0 (2 + 8c^2 - 4c^2 \cos^2\theta_0)\sqrt{1 + c^2 \cos^2\theta_0}$$

$$+ 2N\kappa \sec^3\theta_0 \sqrt{1+c^2} \, \left(1 + 3c^2 - 6c^2 \cos^2\theta_0 + 2c^2 \cos^4\theta_0\right)$$

$$- 2N \sec^4\theta_0 \left(1 - 4c^2 \sin^2\theta_0 - 8c^4 \sin^2\theta_0\right) \log\left(\frac{c + \sqrt{1+c^2}}{c\cos\theta_0 + \sqrt{1 + c^2 \cos^2\theta_0}}\right)$$

$$+ 8N\kappa^3 \sin\theta_0 \tan\theta_0 \sqrt{1+c^2} \Big(2\log\epsilon - 2\log(\cos\theta_0 \sin\theta_0) - \log\left(1 + c^2\right)$$

$$+ \log\left(\cos^2\theta_0 \left(1 + 2c^2\right) + 2\cos\theta_0 \sqrt{1+c^2}\sqrt{1 + c^2 \cos^2\theta_0} + 1\right)\Big). \tag{3.48}$$

The Legendre transform of the gauge field is written as the integral over the total derivative

$$S_{L.T.} = -i\frac{L^2}{2\pi\alpha'} \int_0^{\theta_0 - \epsilon} d\theta \int d\psi \, \Pi \, F_{\theta\psi}$$

$$= 4N\kappa \sin\theta_0 \frac{\sqrt{1+c^2}}{\epsilon} - 2Nc\frac{3 + \kappa^2 + 2c^2}{\sqrt{1+c^2}}, \tag{3.49}$$

and the boundary term for ρ is

$$-P_\rho\Big|_{\theta_0 - \epsilon} = -2N\sin\theta_0(\kappa\sqrt{1+\kappa^2} + \operatorname{arcsinh}\kappa)\frac{\sqrt{1+c^2}}{\epsilon\sqrt{1+\kappa^2}}$$

$$- 4N\sec\theta_0(\kappa\sqrt{1+\kappa^2} + \operatorname{arcsinh}\kappa)\frac{2\kappa^2 - (1 + 4\kappa^2 - \kappa^4)\cos^2\theta_0 - 2c^4}{4(1+\kappa^2)^{3/2}\sqrt{1+c^2}}. \tag{3.50}$$

Finally the regularized Wess-Zumino term turns out to be

$$S_{WZ} = -2N\left(c\sqrt{1+c^2} + \operatorname{arcsinh}c\right) - S_{DBI} - S_{L.T.} + P_\rho. \tag{3.51}$$

Those expressions are much more complicated than the 1/2 BPS case, where those three terms are

$$S_{DBI} = 4N\int d\rho \sinh\rho \cosh u \sinh^3 u,$$

$$S_{WZ} = 4N\int d\rho \sinh\rho \left(\frac{u}{2} - \frac{1}{2}\sinh u \cosh u - \sinh^3 u \cosh u\right), \tag{3.52}$$

$$S_{L.T.} = 4N\int d\rho \sinh\rho \cosh u.$$

The boundary term for ρ, just removes the divergence from the upper limit of ρ integration, giving -1 from the lower limit. Summing up all the contributions and using $\sinh u = \kappa$ gives the full on-shell action [4]

$$S = -2N\left(\kappa\sqrt{1+\kappa^2} + \operatorname{arcsinh}\kappa\right). \tag{3.53}$$

While the regularized expressions for the 1/4 BPS loop are much more complicated, the sum of the bulk and boundary terms is exactly the same with the replacement of κ by c

$$S_{\text{total}} = -2N \left(c\sqrt{1+c^2} + \operatorname{arcsinh} c \right). \tag{3.54}$$

Recall that c is related to the number of units of flux carried by the brane, or the rank of the representation of the Wilson loop by (3.42)

$$c = \kappa' = \frac{k \cos\theta_0 \sqrt{\lambda}}{4N}. \tag{3.55}$$

Therefore this stable solution will contribute to the expectation value of the 1/4 BPS Wilson loop at strong coupling

$$\langle W_{\kappa'} \rangle = \exp\left[2N \left(\kappa'\sqrt{1+\kappa'^2} + \operatorname{arcsinh}\kappa' \right) \right]. \tag{3.56}$$

This is the same result as can be derived from the matrix model observable (either the multiply wrapped loop [4] or the symmetric one [8, 9]) in this limit. This serves as a confirmation that the matrix model correctly captures the 1/4 BPS loop including all $1/N$ corrections at large λ.

An analogous computation can be done for the unstable branch, where the range of integration for the coordinate θ is $[\theta_0 + \epsilon, \pi]$ and P_ρ is evaluated at $\theta_0 + \epsilon$. The final result is exactly as above, except for the overall sign

$$S_{\text{total}}^{(\text{unstable})} = 2N \left(\kappa'\sqrt{1+\kappa'^2} + \operatorname{arcsinh}\kappa' \right). \tag{3.57}$$

As in the case of the string solution reviewed before, this should correspond to an exponentially small correction to the expectation value of the Wilson loop when doing the asymptotic expansion at large N and large λ (see the discussion after eq. (3.6)).

In addition to the 1/2 BPS limit, with $\theta_0 = 0$, there is another interesting limiting case, of $\theta_0 = \pi/2$ studied by Zarembo [18]. In that case the two D3-brane solutions, whose actions always have the opposite signs, are degenerate. Both have vanishing action, and in fact there are more than two solutions, rather a whole family parameterized by an S^3. But, unfortunately, looking at the solutions at this limit we find that they do not provide a good description for the Wilson loop. If we consider finite κ, then from (3.42), the constant c vanishes and by (3.34), also $u = 0$. Therefore the D3-brane shrinks to a two-dimensional surface and therefore the higher-derivative corrections to the DBI action cannot be ignored.

If instead we keep c finite in that limit, then κ will diverge, leading to a smooth D3-brane solution. But now as θ goes to θ_0 both ρ and u diverge, meaning that the brane ends along a 3-dimensional surface on the boundary, rather than the Wilson loop.

4. Discussion

We have presented some solutions for D3-branes in $AdS_5 \times S^5$, which are dual to certain 1/4 BPS Wilson loop operators in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. The first example was a combined system of a loop with two local insertions made from complex

scalar fields. Without the insertions the loop itself would have been 1/2 BPS and the trace of the local insertions is also 1/2 BPS, while the combined system preserves 1/4 of the supersymmetries. The second system was a family of Wilson loops with couplings to three of the scalars in a way that also preserves eight supercharges.

It is by now a standard feature of the AdS/CFT correspondence that very long operators in the gauge theory map to "giant" D-brane objects rather than to fundamental strings or supergravity modes. In our case the D3-branes should describe the Wilson loops in a high-dimensional symmetric representation, where the rank of the representation k is of order N. In the example of the loop with insertions we were able to calculate the energy and angular momentum and they agreed with each other, as would be expected, but there was no special feature arising from the fact that the loop is in a certain representation.

In the second example we were able to compare the result of the AdS calculation to a matrix model conjectured to describe those 1/4 BPS loops [22, 23, 19]. The value of those loops at large N and large λ in a symmetric representation is known (and coincides with the single-trace multiply-wrapped loop). We found that the classical action for the D3-brane correctly reproduces the expected result, which includes an infinite series of $1/N$ corrections to the planar string expression. Furthermore, we have found two solutions with the same boundary conditions, in exact analogy with the strings describing the loop in the fundamental representation. The second solution, which contributes an exponentially small correction to the Wilson loop in the supergravity limit is the brane analog of a world-sheet instanton. Such contributions are expected, since the string expansion is asymptotic in $1/\sqrt{\lambda}$.

The geometry of this second D3-brane solution is very interesting. Starting from the boundary of AdS, where it originates along the Wilson loop, it moves into the bulk, turns back, goes again to the boundary, gets reflected back into the interior, and closes off smoothly on itself. If we chose not to continue the solution, it would end on a two-dimensional surface on the boundary. So this part of the solution would describe the correlator of a Wilson loop and a surface operator which are non-trivially linked. It would be very interesting to understand further the nature of this surface operator. The connection between Wilson loops and surface operators may not be so surprising given that they may both be described by branes in the bulk (see *e.g.* [26, 7, 27]).

As discussed at the end of the last section, one would like also to consider a special limit of these loops, when $\theta_0 = \pi/2$. This limit is particularly interesting because of a comment made at the end of [19], where it was noticed that one may take λ large while keeping $\lambda' = \lambda \cos \theta_0$ small, in a way similar to the BMN limit [28]. When considering the string solution in that limit, the mass of the string modes becomes much larger than the mass of the three broken zero modes (those parameterizing the S^3 mentioned above). Ignoring all the stringy modes and integrating only over those three leads to the full result of the planar matrix model, including all α' (or $1/\sqrt{\lambda'}$) corrections. It would be extremely interesting if we were able to repeat the calculation here and find the exact expression including all $1/N$ and $1/\sqrt{\lambda'}$ corrections. Recall that those corrections would not be the same for the loop in the symmetric representation and for the multiply-wound loop. So this calculation would be a very good check of the recent identification of the D3-brane

with the loop in the symmetric representation [10]. Unfortunately, as explained before, in this limit the D3-brane degenerates and does not provide a good description of those Wilson loops.

The loops studied in this paper are not the most general 1/4 BPS Wilson loops, all our examples had a circular geometry, which is not required. Many other loops were described in [18], and there are probably even more. The string solutions describing those loops were studied by Dymarsky et al. [29], and perhaps there is a general classification of the relevant branes along the lines of [30].

After studying the probe brane in the $AdS_5 \times S^5$ background it is natural to consider the back-reaction of the brane on the geometry. This was pursued in the 1/2 BPS case in [31] and [32], where all the relevant metrics could be related to Young-tableaux, thus giving a correspondence between the representations of the Wilson loop and the associated metrics. It would be interesting to try to find the metrics in this case too, though this system has far less symmetry making it a much harder problem.

Finally, one can go further to a system which is only 1/8 BPS, by looking at the correlators of those Wilson loops with chiral primary local operators. Amazingly, in the gauge theory those also seem to be captured fully by ladder diagrams and may be reduced to some matrix model [24]. In the case of the 1/2 BPS loop this was checked in AdS using a string [33, 34] and D-branes [35]. For the 1/4 BPS loop this was done with a string in [24] and would be interesting to repeat this calculation with D3-branes.

Acknowledgments

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